Brownian Motion.

Recall: a stochastic process $\{X(t) \mid t \in T\}$ is a collection of random variables $X(t)$ for each value (parameterized by) of $t \in T$. These random variables are functions from probability space Ω to S (the set of states). In particular, we can think of the stochastic process as a f-n of two arguments $\omega \in \Omega$ and $t \in T$:

$$X: \Omega \times T \to S.$$

Def-n. Let us consider an $\omega \in \Omega$ and fix it. Then the f-n $X_\omega(t) := X(\omega, t)$ is called a path (trajectory) of the stochastic process.

Example (random walk). Let $T = \mathbb{Z}_{\geq 0}$, $S = \mathbb{Z}$ and

$X : \Omega \times T \to S$, i.e. $X : \Omega \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ given by

$X(\omega, 0) = 0$ for any $\omega \in \Omega$ and for $n \geq 1$

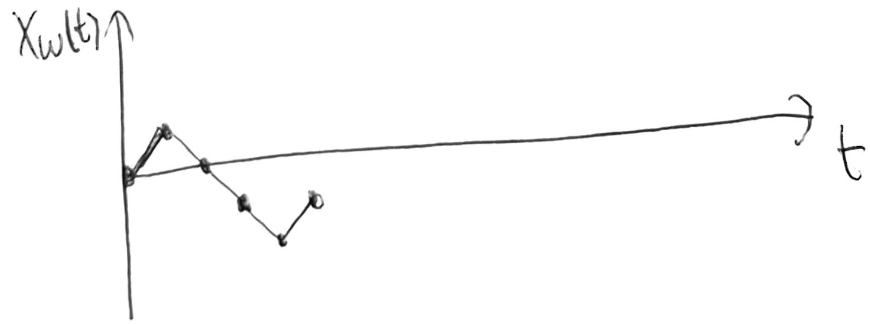
$$X(\omega, n) = \sum_{i=1}^{n} \xi_i(\omega),$$ where $\xi_1, \dots, \xi_n$ are i.i.d.

random variables with mean 0 and variance 1.

Consider $\omega \in \Omega$, s.t. $\xi_1(\omega) = 1$, $\xi_2(\omega) = -1$, $\xi_3(\omega) = -1$, $\xi_4(\omega) = -1$, $\xi_5(\omega) = 1$, ...

Then $X_\omega(1) = 1$, $X_\omega(2) = 1 - 1 = 0$, $X_\omega(3) = 1 - 1 - 1 = -1$,

$X_\omega(4) = X_\omega(3) - 1 = -2$, $X_\omega(5) = X_\omega(4) + 1 = -1$, ...

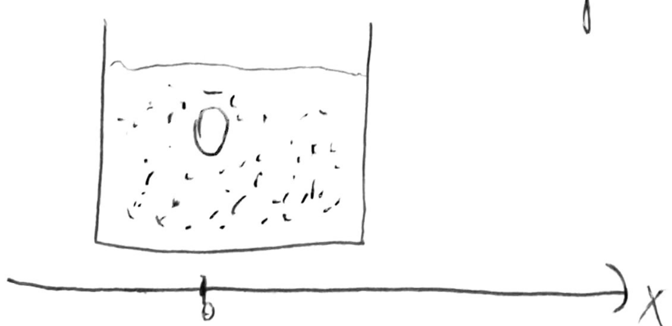


Def-n. A standard Brownian motion (a standard Wiener process) is a stochastic process $\{B(t) \mid t \in \mathbb{R}_{\geq 0} = [0; +\infty)\}$ with the following properties:

1. $B(0) = 0$ (with probability 1);
2. the functions $B_\omega(t)$ are continuous (with probability 1) in t for all $\omega \in \Omega$.
3. the process $\{B(t) \mid t \in \mathbb{R}_{\geq 0}\}$ has stationary independent increments, i.e. for any collection of non overlapping intervals $[s_i, t_i]$, $i = 1, \dots, n$ the random variables $B(t_i) - B(s_i)$ are mutually independent;
4. the increment $B(t) - B(s)$ has normal distr-n with mean 0 and variance $t-s$, i.e. $B(t) - B(s) \sim N(0, t-s)$ (here $t > s \geq 0$).

Rmk: property (2) implies memorylessness.

Example 1. Consider a particle in water. For simplicity we will only take into account the movements in one direction ('left' and 'right').



The number of collisions of our particle with other molecules is about 10^{14} per second. Those collisions are random (occur at all directions) and cause the particle to jiggle. The model that describes the movement of the particle along the x-axis is Brownian motion.

This allowed Einstein to determine the size of an atom (the number of molecules contained in a gram-molecule), see 'Investigations on the Theory of Brownian Movement', 1905.

Example 2. Five years before Einstein's discovery a French mathematician L. Bachelier applied Brownian motion to describe stock and option market price fluctuations. This was done in his PhD thesis under the supervision of Henri Poincaire. The key insight of his thesis was that if asset prices in the short term show an identifiable pattern, speculators will find this pattern and exploit it, thereby eliminating it.

Rmk (Brownian motion as the limit of a random walk).
Construction (sketch):
1. Divide the interval $[0,+\infty)$ into subintervals of length δ: $[0;+\infty) = \{0\} \cup (0;\delta] \cup (\delta;2\delta] \cup (2\delta;3\delta] \cup \ldots$
2. Associate to each subinterval $((k-1)\delta;k\delta]$ a random variable $X_k = \begin{cases} \sqrt{\delta}, & p=1/2 \\ -\sqrt{\delta}, & p=1/2. \end{cases}$

The X_k's are independent, identical distr-ns with $\mathbb{E}(X_k)=0$, $Var(X_k)=\delta$.
3. For $t_n = n\delta$, $n \in \mathbb{Z}_{\geq 0}$ introduce the random variable $\tilde{B}(t_n) = \sum_{k=1}^{n} X_k$. Notice that $\mathbb{E}(\tilde{B}(t_n)) = n\mathbb{E}(X_1)=0$ and $Var(\tilde{B}(t_n)) = n Var(X_1) = n\delta = t_n$. Set $\tilde{B}(0)=0$.

When $\delta \to 0$,
we get that each interval $(s;t]$ is subdivided into more and more subintervals of length δ. Using the CLT we show that $B(t)-B(s) \sim N(0,t-s)$ (property (4)). It is straightforward to check properties (2) and (3) as well.

Exercise. Show that with probability one a path $X_\omega(t)$, where $\{X(t) \mid t \in \mathbb{Z}_{\geq 0}\}$ is the random walk described in example on page 1, visits every given integer $l \in \mathbb{Z}$.

Corollary. With probability one, the Brownian motion visits every real number.

Fact. Almost surely (with probability 1) the Brownian motion is nowhere differentiable, i.e. $\forall t_0 \in [0;+\infty)$ $P(\omega \in \Omega \mid B_\omega(t) \text{ is differentiable at } t_0) = 0$.

Thm. $(dB(t))^2 = dt$. (This is the analogue of $Var(\tilde{B}(t)) = t$).

Corollary (of the fact): the calculus we are used to does not apply to f-ns of $B(t)$. We need some different framework...

Ito's calculus (very brief overview).

Idea. Let $f(t, B_t)$ be a smooth function of two variables. Then $\frac{df(t, B(t))}{dt} = f'_{B(t)}(t, B(t)) \cdot \frac{dB(t)}{dt} + f'_t(t, B(t))$.

We have a problem: $\frac{dB(t)}{dt}$ does not exist.

Let us try to find an expression for $df(t,B(t))$.
We will use the Taylor series:

$$df(t,B(t)) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial B(t)}dB(t) + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial B(t)^2}dB(t)^2 + \frac{\partial^2 f}{\partial t\,\partial B(t)}dt\,dB(t) + \frac{\partial^2 f}{\partial t^2}dt^2\right) + \ldots$$

As in standard calculus we would like to consider the terms up to $o(dt)$ and due to the Fact, we get

$$\boxed{df(t,B(t)) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial B(t)^2}\right)dt + \frac{\partial f}{\partial B(t)}dB(t)}$$

<u>Lemma (Itô)</u>. Let $f(t,x)$ be a smooth f-n of two variables and $X(t)$ a stochastic process satisfying $dX(t) = \mu(t)dt + \sigma(t)dB(t)$, where $B(t)$ is a Brownian motion. Then

$$df(t,X(t)) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X(t)}\cdot dX(t) + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial X(t)^2}dX(t)^2 =$$

$$= \left(\frac{\partial f}{\partial t} + \mu(t)\frac{\partial f}{\partial X(t)} + \frac{\sigma^2(t)}{2}\cdot\frac{\partial^2 f}{\partial X(t)^2}\right)dt + \sigma(t)\cdot\frac{\partial f}{\partial X(t)}B(t).$$

Examples:

(1) $f(t, B(t)) = \frac{B(t)^2}{2}$

$df(t, B(t)) = ?$

$\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial B(t)} = B(t)$, $\frac{\partial^2 f}{\partial B(t)^2} = 1$, so $df(t, B(t)) = \frac{1}{2}dt + B(t)dB(t)$

(2) geometric Brownian motion.

$f(t, B(t)) = e^{\sigma B(t) + \mu t}$

$\frac{\partial f}{\partial t} = \mu \cdot f(t, B(t))$, $\frac{\partial f}{\partial B(t)} = \sigma f(t, B(t))$, $\frac{\partial^2 f}{\partial B(t)^2} = \sigma^2 f(t, B(t))$

Hence, $df(t, B(t)) = (\mu + \frac{\sigma^2}{2}) f(t, B(t)) dt + \sigma \cdot f(t, B(t)) dB(t)$.

Rmk: if $\mu = -\frac{\sigma^2}{2}$, we get $df = \sigma \cdot f(t, B(t)) dB(t)$.

Call and Put Options.

Def-n. An option is a right to buy or sell a financial asset at a fixed price (at a fixed time).

Options

Call
gives the owner the right to buy the asset at a certain price

Put
gives the owner the right to sell the asset at a certain price

Rmk: as options offer a beneficial right to do something (buy or sell), they cost money.

Examples.

(1) Yesterday (July 28^{th}) the call option for a share of Macy's stock with strike price \$6.00 and execution date July 31^{st} was priced at \$0.48

(2) the put option with strike price \$6.50 and same execution was priced at \$0.16.

Let S denote the current price of an asset, E the strike price and t, T the current and execution times, respectively, then

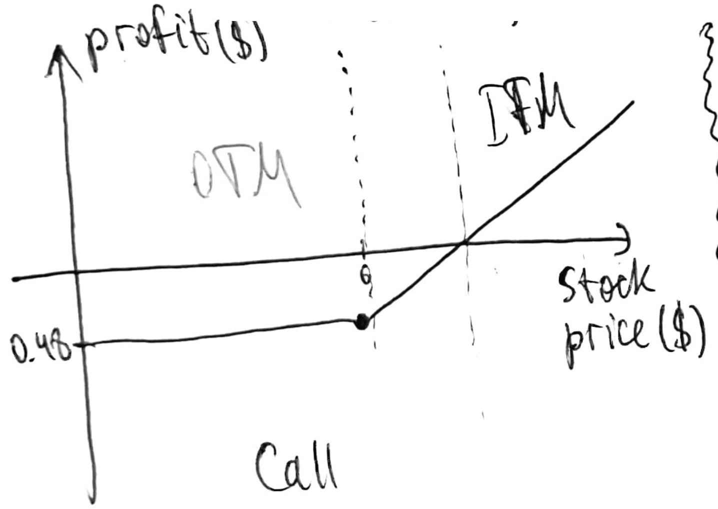
$$\text{Profit from } Call_E(S,t) = \max(0, S-E) - \text{price of the option}$$

$$\text{Profit from } Put_E(S,t) = \max(0, E-S) - \text{price of the option.}$$

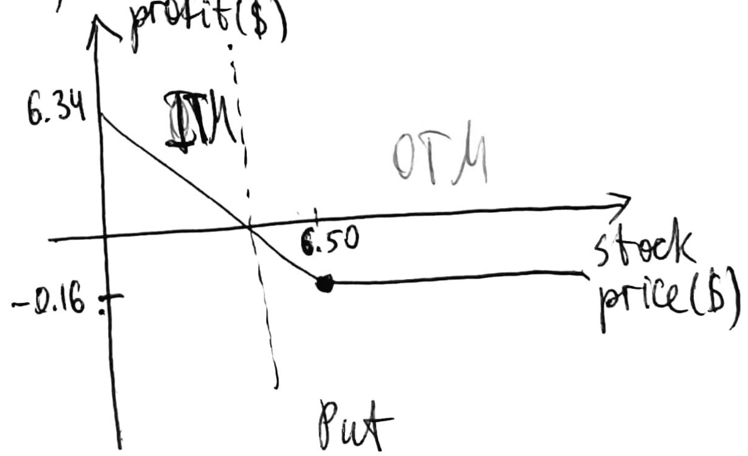
In our examples:

$$\max(0, S-6) - 0.48$$

$$\max(0, 6.50-S) - 0.16$$



Call



Put

If $Call_E(S,t) > 0$, we say that the Call (Put) option ($Put_E(S,t) > 0$)

is 'in the money' (ITM), while when $Call_E(S,t) < 0$, $Put_E(S,t) < 0$, 'out of the money' (OTM).

Designing Portfolios.

Def-n. A portfolio is a collection of financial assets (stocks, bonds, options, cash, etc.) held by an individual, investment company or financial institution.

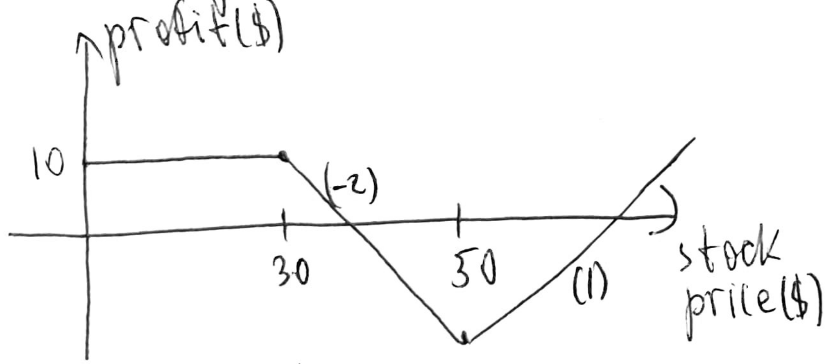
We will focus on portfolios containing Calls, Puts and cash.

Def-n. To go short on a Put (Call) means to sell the corresponding option.

Suppose we would like to draw the profit curve (graph profit as a f-n of stock price) given the portfolio and vice versa.

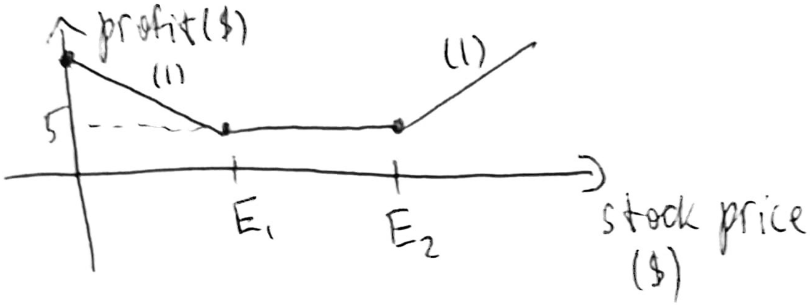
Examples: draw the profit curves at $t=T$:

1. Consider the portfolio $P(t) = -2C_{30}(S,t) + 3C_{50}(S,t) + \$10$



Profit curve.

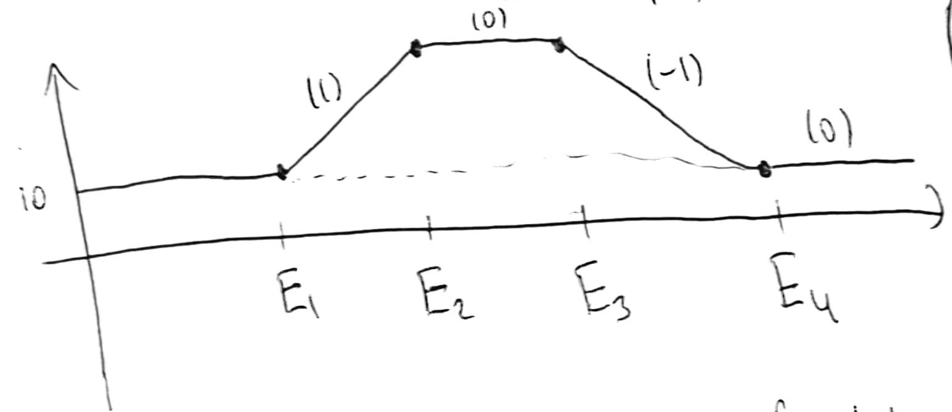
(2) $P = P_{E_1} + C_{E_2} + \$5$ for $E_1 < E_2$.



'Long Straddle'

(3) $P = \$10 + C_{E_1} - C_{E_2} - C_{E_3} + C_{E_4}$, where $E_1 < E_2 < E_3 < E_4$.

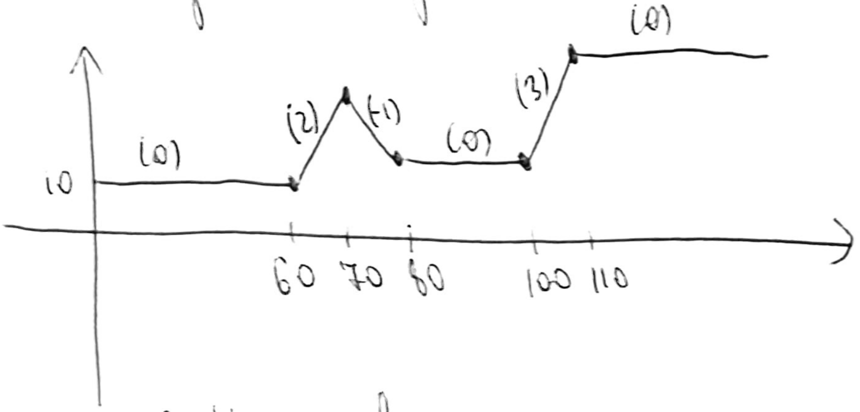
(assume $E_4 - E_3 = E_2 - E_1$)



'Long Condor'.

<u>Examples:</u> For a given shape of the profit curve design the portfolio:

(1)



(a) using Calls only:

$P = \$10 + 2C_{60} - 3C_{70} + C_{80} + 3C_{100} - 3C_{110}$ (go 'from left to right')

(b) using Puts only: $-3P_{110} + 3P_{100} + P_{80} - 3P_{70} + 2P_{60} + \$10$

(go 'from right to left')

Put-Call Parity Equation.

We start with recalling the formula for the interest rate compounded contiously in a Bank: if the initial deposit was D, then after time t it will become De^{tr}, where r stands for the rate.

Suppose we would like to solve the reverse problem: how much money do we need to deposit at time t, so that the money at time T is E?

Answer: we get the equation $D \cdot e^{(T-t)r} = E$, which allows to find $D = Ee^{-r(T-t)}$.

Thm. Consider an asset (stock) with Put and Call options having the same strike price E and executoin time T. Then the following equation holds:

$$\boxed{S + Put_E(S,t) = Call_E(S,t) + Ee^{-r(T-t)}}$$

Rmk. We assume there can be no arbitrage opportunity (risk-free profit).

Proof: First we check that the equality holds at time $t = T$. For this consider two cases:

1. $S(T) \geqslant E$, then $Put_E(S,T) = \max(0, E-S) = 0$ and $Call_E(S,T) = \max(0, S-E) = S - E$

Hence, we get $S + 0 = S - E + Ee^{-r(T-T)}$ or $S = S$ ✓

2. $S(T) < E$, then $Put_E(S,T) = E - S$ and $Call_E(S,T) = 0$, so

$$S + E - S = 0 + E \text{ or } E = E \checkmark$$

Now let $t < T$ and assume the equality does not hold. Suppose $S + Put_E(S,t) > Call_E(S,t) + Ee^{-r(T-t)}$ (*).

We will show that (*) leads to an arbitrage opportunity.

To create a risk-free profit:

1. go short on K puts and K stock assets;
2. buy K calls and put $KEe^{-r(T-t)}$ to the bank;
3. keep the change $(L = K(S + Put_E(S,t) - Call_E(S,t) - Ee^{-r(T-t)}) > 0)$

As we have showed that at time $t = T$ the equality holds, a risk-free profit of \$L was made.

<u>Exercise</u>. Show how to make a risk-free profit in case $S + Put_E(S,t) < Call_E(S,t) + Ee^{-r(T-t)}$.

<u>Example</u>. Suppose we know that for a share of stock of a certain asset $C_{50}(51,t) - P_{50}(51,t) = 7$ with $T-t = 1$ year and interest rate $r = 0.02$ in the bank. How can we use this information to our advantage?

Answer: Let $P_{50}(51, t) = X$, then $C_{50}(51, t) = X+7$, so

$$S + Put_E(S,t) = 51+X,$$ while

$$Call_E(S,t) + Ee^{-r(T-t)} = X+7+50 \cdot e^{-0.02 \cdot 1} \simeq X+7+49$$

Hence, $S + Put_E(S,t) = 51+X \cancel{<} X+56 = Call_E(S,t) + Ee^{-r(T-t)}$,

so there is an arbitrage opportunity.